Analysis 2

26 March 2024

Warm-up: calculate $f''_{yx} = (f'_y)'_x$

$$f(x, y) = 3\cos(2y) + y^2 \ln(x).$$

Happy Holidays

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
☪️ First day of Ramadan	Lecture	Problem Session				☘️ St. Patrick's Day
	Lecture	Problem Session				✡️ Purim
🕉️ Holi	Lecture (today)	Problem Session				✝️ Easter
		Problem Session				

Directional derivative

The directional derivative of $f(x, y)$ at the point (a, b) in the direction of the unit vector $\hat{u} = [u_1, u_2]$ is

$$f'_{\hat{u}}(a, b) = \nabla f(a, b) \cdot \hat{u}$$

- This formula only works when $\hat{u}$ a *unit* vector!
- For the direction of any vector $\vec{v}$, use $\hat{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{v}}{\sqrt{v_1^2 + v_2^2}}$.

At the point (a, b), the directional derivative $f'_{\hat{u}}(a, b)$ is largest when $\hat{u}$ points in the same direction as $\nabla f(a, b)$.

Critical points

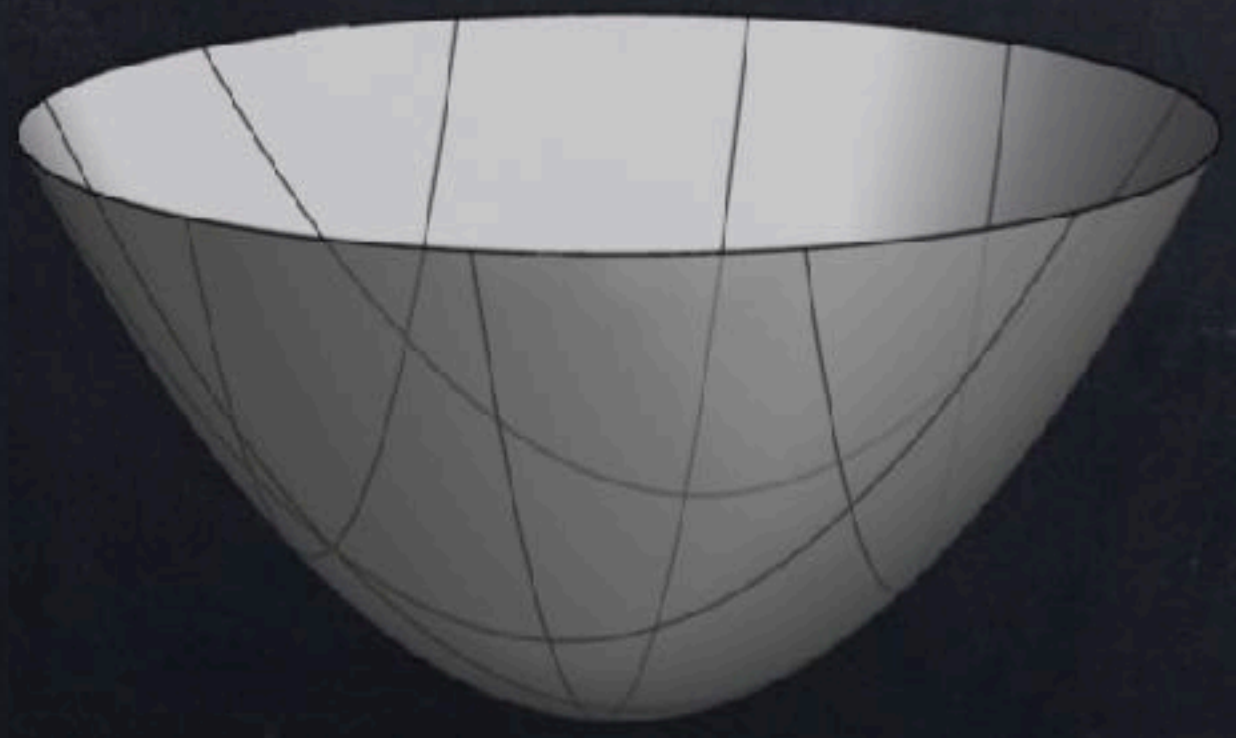
Last time

A **critical point** of $f(x, y)$ is a point (x, y) where ∇f is zero or undefined.

- Note "zero" is $0\hat{\imath} + 0\hat{\jmath}$, so "$\nabla f = \vec{0}$" means $f_x(x, y) = 0$ AND $f_y(x, y) = 0$.
- We have to solve a system of equations to find the CP of $f(x, y)$!
  - This is usually not a linear system, so we cannot use Gaussian elimination or matrix inverse. It's also very hard to know at the start how many solutions there will be.

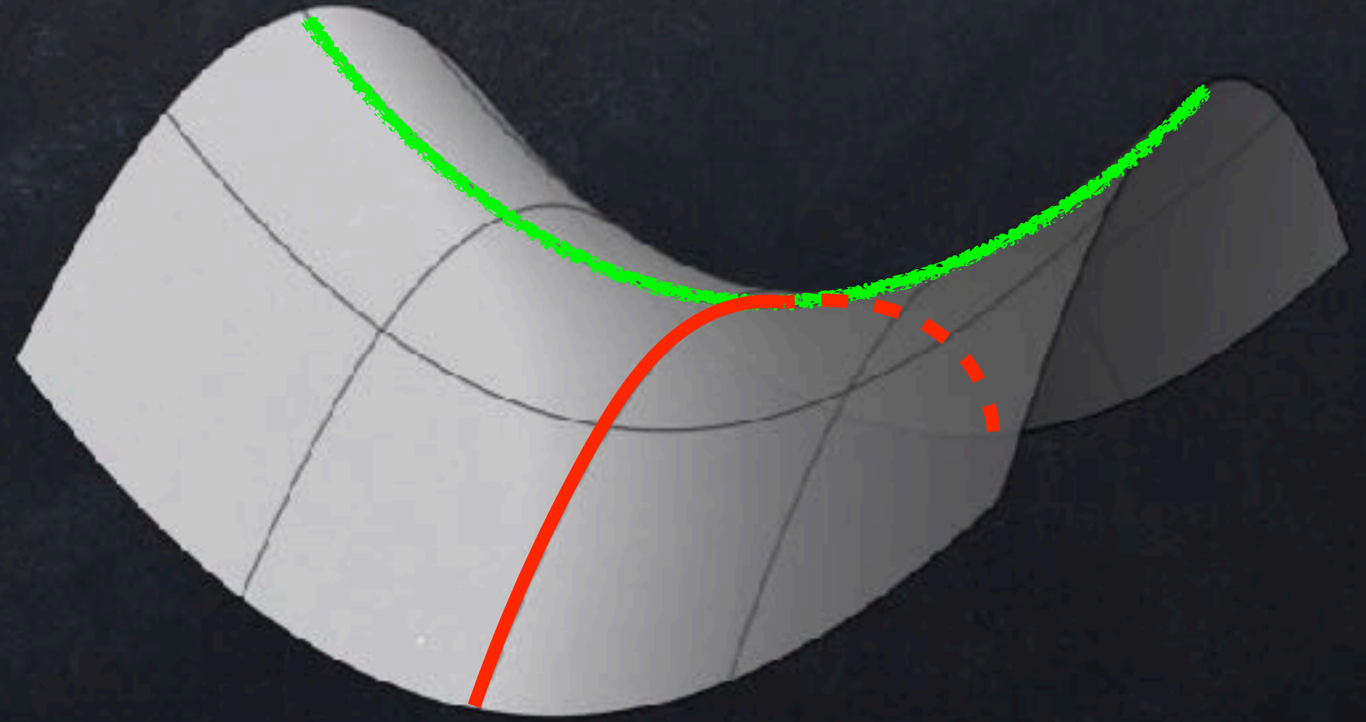
How can we classify critical points?

New



local min

local max



saddle

Second Derivative Test

Analysis 1

1. To find the critical points of $f(x)$: solve $f'(x) = 0$ or undefined.

2. To classify a critical point of $f(x)$:

If $f'' > 0$ then the CP is a LOCAL MIN.

If $f'' < 0$ then the CP is a LOCAL MAX.

If $f'' = 0$ then the test does not tell you what kind of CP this is.

In An. 1 there is also the First Deriv. Test. We do *not* have this in Analysis 2.

Second Derivative Test

Analysis 2

1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

2. To classify a critical point of $f(x, y)$:

If $f''_{xx} f''_{yy} - (f''_{xy})^2 > 0$ and $f''_{xx} > 0$ then the CP is a LOCAL MIN.

If $f''_{xx} f''_{yy} - (f''_{xy})^2 > 0$ and $f''_{xx} < 0$ then the CP is a LOCAL MAX.

If $f''_{xx} f''_{yy} - (f''_{xy})^2 < 0$ then the CP is a SADDLE.

If $f''_{xx} f''_{yy} - (f''_{xy})^2 = 0$ then the test does not tell you what kind of CP this is.

You could just memorize this, but it may help to know WHY.

First derivatives

There are at least four kinds of "first derivatives" for $f(x, y)$:

- partial derivative with respect to x,
- partial derivative with respect to y,
- gradient,
- directional derivative.

There are several kinds of second derivatives for $f(x, y)$ also.

Second derivatives

There are several kinds of second derivatives for $f(x, y)$:

- second partial derivative with respect to x

$$f''_{xx} = (f'_x)'_x = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2},$$

- second partial derivative with respect to y

$$f''_{yy} = (f'_y)'_y = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2},$$

- mixed partial derivatives,
- Hessian.

Second derivatives

There are several kinds of second derivatives for $f(x, y)$:

- second partial derivative with respect to x
- second partial derivative with respect to y
- mixed partial derivatives,

$$f''_{xy} = (f'_x)'_y = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$f''_{yx} = (f'_y)'_x = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x \partial y},$$

- Hessian.

Second derivatives

There are several kinds of second derivatives for $f(x, y)$:

- second partial derivative with respect to x,
- second partial derivative with respect to y,
- mixed partial derivatives,
- Hessian

$$\mathbf{H}f = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}.$$

This is a <u>matrix</u> (similar to how ∇f is a vector).

Example: Calculate all four second derivatives for $f(x, y) = ye^{2x+8y}$.

- $f''_{xx} =$
- $f''_{yy} =$
- $f''_{xy} =$
- $f''_{yx} =$

Symmetry of second derivatives

If the second derivatives of $f(x, y)$ are continuous, then

$$f''_{xy} = f''_{yx}.$$

Task: Calculate $\nabla f(-4, 1)$ and $\mathbf{H}f(-4, 1)$ for $f(x, y) = ye^{2x+8y}$.

Second Derivative Test

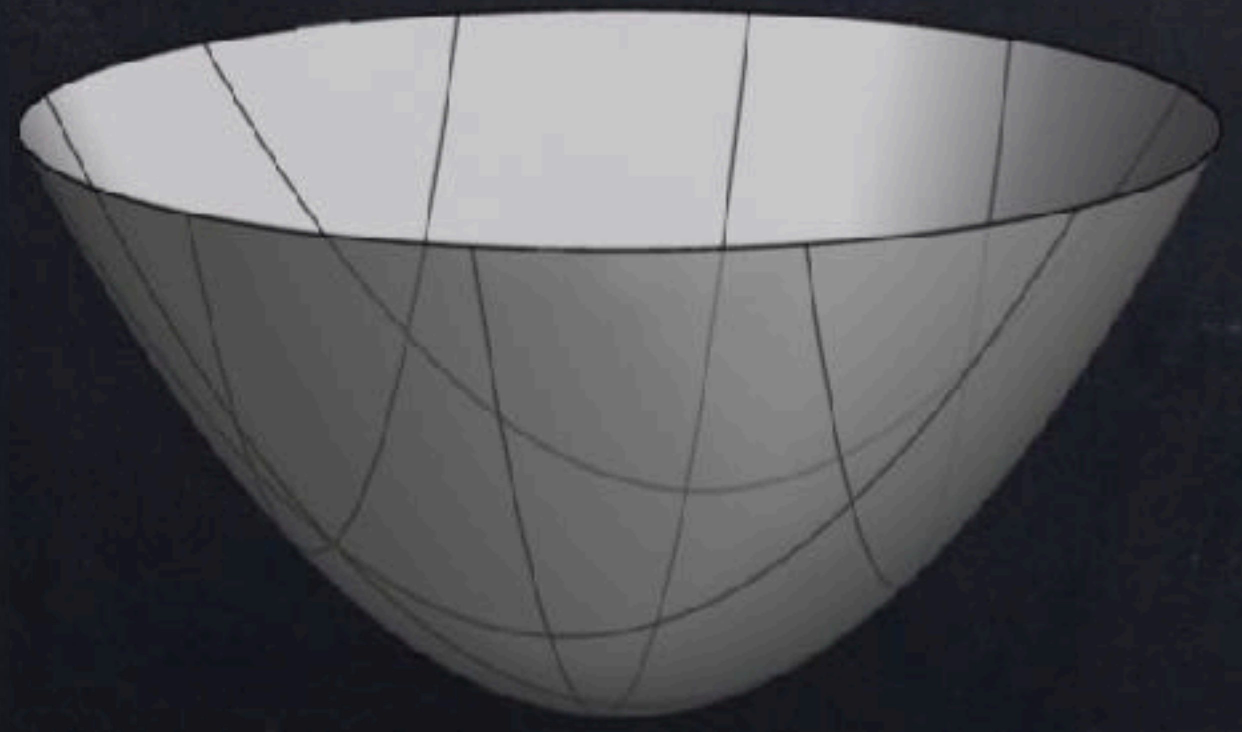
1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

To classify the critical points:

v0

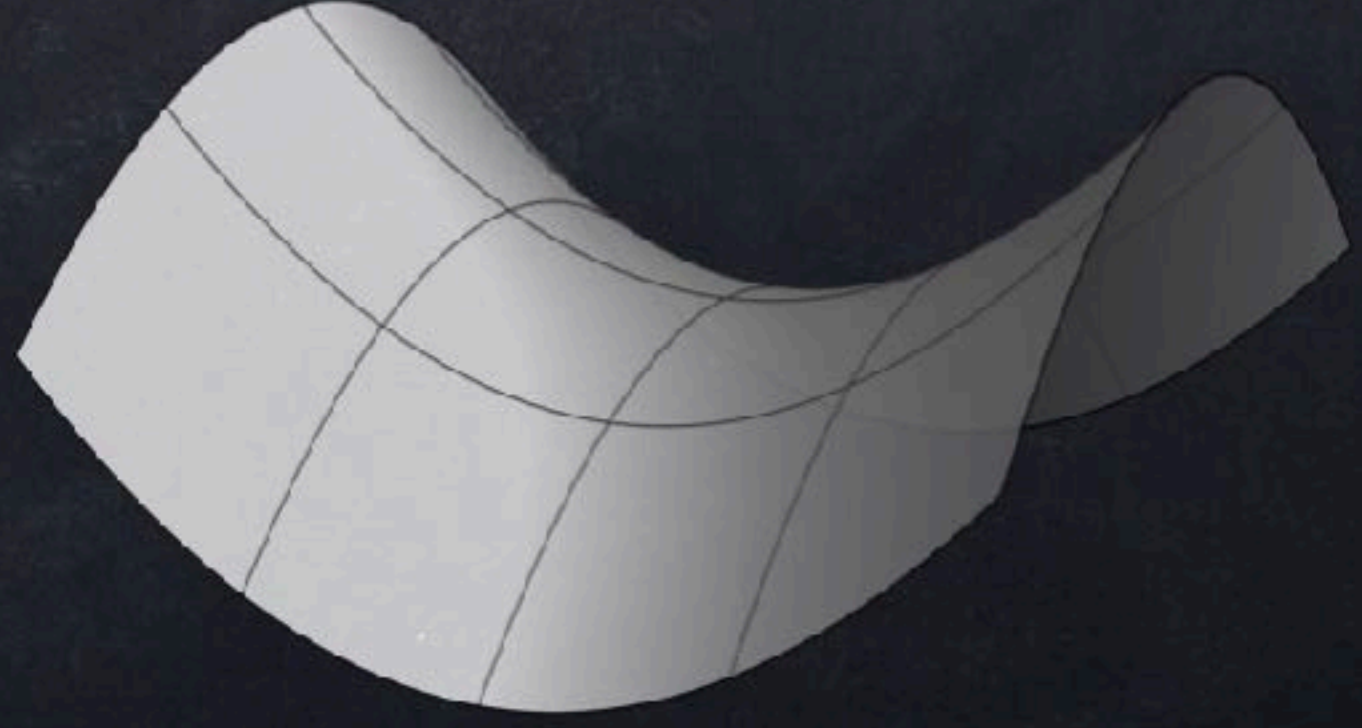
2. Compute $\mathbf{H}f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$ at each CP.
3. If $\det(\mathbf{H}f) > 0$ then the CP is either a LOCAL MAX or LOCAL MIN.

   If $\det(\mathbf{H}f) < 0$ then the CP is a SADDLE.



local min

local max



saddle

Second Derivative Test

1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

To classify the critical points:

v1

2. Compute $\mathbf{H}f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$ at each CP and let λ_1, λ_2 be its eigenvalues.
3. If $\lambda_1, \lambda_2 > 0$ then the CP is a LOCAL MIN.
   If $\lambda_1, \lambda_2 < 0$ then the CP is a LOCAL MAX. } det > 0
   If λ_1, λ_2 have different ± signs then the CP is a SADDLE. } det < 0

Algebra facts: for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

- $\det(M) = \lambda_1 \times \lambda_2$.
- If $\det(M) > 0$ and $a > 0$ then λ_1 and λ_2 are both positive.
- If $\det(M) > 0$ and $a < 0$ then λ_1 and λ_2 are both negative.

Second Derivative Test

1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

To classify the critical points:

v2

2. Compute $\mathbf{H}f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$ at each CP.
3. If $\det(\mathbf{H}f) > 0$ and $f''_{xx} > 0$ then the CP is a LOCAL MIN.
   If $\det(\mathbf{H}f) > 0$ and $f''_{xx} < 0$ then the CP is a LOCAL MAX.
   If $\det(\mathbf{H}f) < 0$ then the CP is a SADDLE.

You can check f''_{yy} instead (it will have the same sign as f''_{xx} if det>0).

If $\det(\mathbf{H}f) = 0$ then the test doesn't tell us what kind of CP this is.

Second Derivative Test

1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

To classify the critical points:

v3

2. Compute f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy} at each CP.

3. If $f''_{xx}f''_{yy} - (f''_{xy})^2 > 0$ and $f''_{xx} > 0$ then the CP is a LOCAL MIN.
   If $f''_{xx}f''_{yy} - (f''_{xy})^2 > 0$ and $f''_{xx} < 0$ then the CP is a LOCAL MAX.
   If $f''_{xx}f''_{yy} - (f''_{xy})^2 < 0$ then the CP is a SADDLE.

You can check f''_{yy} instead (it will have the same sign as f''_{xx} if det>0).

If $f''_{xx}f''_{yy} - (f''_{xy})^2 = 0$ then the test doesn't tell us what kind of CP this is.

Example: Find and classify the critical points of $x^3 - 3xy + 8y^3$.

Last week: $\begin{cases} 3x^2 - 3y = 0 \\ 24y^2 - 3x = 0 \end{cases} \rightarrow (0, 0)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$

x	y	D	f_{xx}''	Type
0	0	?	?	?
1/2	1/4	?	?	?

Task: Find and classify the critical points of $x^2 + 8y^2 - xy^2$.

x	y	D	f_{xx}''	Type